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1987 J. Phys. A: Math. Gen. 20 L769

(http://iopscience.iop.org/0305-4470/20/12/004)

J. Phys. A: Math. Gen. **20** (1987) L769–L770. Printed in the UK

LETTER TO THE EDITOR

Amplitude–exponent relation for the correlation length in the (2+1)-dimensional Ising model

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Received 11 May 1987

Abstract. For the critical (2+1)D Ising model defined on a square lattice with antiperiodic boundary conditions, numerical studies suggest relations among the critical exponents and the finite-size scaling amplitudes of correlation lengths similar to those obtained in (1+1) dimensions from conformal invariance.

For two-dimensional systems defined on an infinitely long strip of finite width N and periodic boundary conditions, it was shown that the hypothesis of conformal invariance at the critical point yields a relationship between the finite-size scaling amplitude A_i of the inverse correlation length $\xi_i^{-1} = A_i N^{-1}$ and the bulk critical exponent x_i (Cardy 1984, von Gehlen *et al* 1986)

$$A_i = 2\pi x_i. \tag{1}$$

We raise the question of whether a linear amplitude-exponent relation like (1) is possible for *three-dimensional* models.

As an example, we study the (2+1)D Ising model, defined on a square $N \times N$ lattice. The Hamiltonian is (Henkel 1984, 1986, 1987)

$$H = -h\sum_n \sigma^z(n) - \tfrac{1}{2}\sum_{(n,n')}[(1+\eta)\sigma^x(n)\sigma^x(n') + (1-\eta)\sigma^y(n)\sigma^y(n')] \tag{2}$$

where nearest-neighbour interactions are understood and the σ^x, σ^y, σ^z are the Pauli matrices. Since, for $\eta = 1$, H is the Hamiltonian limit of the transfer matrix of the 3D Ising model, we have a geometry which is infinite in one direction but finite in the other two directions. For each $\eta \neq 0$, there is a critical point falling into the 3D Ising universality class. Both periodic and antiperiodic boundary conditions will be studied.

H commutes with the operator

$$Q = \frac{1}{2}\left(1 - \prod_n \sigma^z(n)\right) \tag{3}$$

and the corresponding eigenspaces are called sector 0 and sector 1, according to the eigenvalues of Q. The inverse spin-spin correlation length ξ_0^{-1} is the energy gap between the two lowest lying states in the two sectors and the inverse energy-energy correlation length ξ_ε^{-1} is the gap between the ground state and the first excited state in sector 0.

Since one does not leave the 3D Ising universality class by varying η, one can check explicitly the universality of the quantities considered. In a previous letter, it was already shown that the ratio A_ε/A_σ of the scaling amplitudes does not depend on η (Henkel 1986), in agreement with the Privman–Fisher (1984) hypothesis.

Table 1. Ratio of the finite-size scaling amplitudes A_ε/A_σ for periodic and antiperiodic boundary conditions for some values of the (irrelevant) coupling η.

η	1.0	0.9	0.7	0.5	0.3
Periodic	3.64	3.64	3.66	3.67	3.5
Antiperiodic	2.76	2.78	2.70	2.80	2.77

In table 1, we give the ratio A_ε/A_σ, computed at the critical point $h_c(\eta)$, for both periodic and antiperiodic boundary conditions. Taking the mean values, we have $A_\varepsilon/A_\sigma = 3.62$ (7) for periodic and $A_\varepsilon/A_\sigma = 2.76$ (4) for antiperiodic boundary conditions, respectively. The numbers in brackets give the uncertainty in the last digit.

In order to check for a linear relation $A \sim x$, we have to compare with $x_\varepsilon/x_\sigma = 2.76$ (2) (see Henkel 1987 and references therein). We can conclude that a relation like equation (1) between the *bulk* critical exponent and the finite-size scaling amplitude of the correlation length is ruled out for periodic but is strongly supported for *antiperiodic* boundary conditions. Since the normalisation of the Hamiltonian is arbitrary, we cannot determine the proportionality constant in $A \sim x$.

Investigations in other models will show whether the conjecture presented is specific to the Ising model or general. It was argued by Cardy (1985) that (1) holds, provided the Hamiltonian is defined on the surface of a sphere rather than on a torus, as in the present work.

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